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Lower Bound of Minimal Time Evolution in Quantum Mechanics

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Abstract We show that the total time of evolution from the initial quantum state to final quantum state and then back to the initial state, i.e., making a round trip along the great circle over S^2, must have a lower bound in quantum mechanics, if the difference between two eigenstates of the 2×2 Hamiltonian is kept fixed. Even the non-hermitian quantum mechanics can not reduce it to arbitrarily small value. In fact, we show that whether one uses a hermitian Hamiltonian or a non-hermitian, the required minimal total time of evolution is same. It is argued that in hermitian quantum mechanics the condition for minimal time evolution can be understood as a constraint coming from the orthogonality of the polarization vector $\mathbf{P}$ of the evolving quantum state $\rho = \frac{1}{2}(\mathbf{1} + \mathbf{P} \cdot \boldsymbol{\sigma})$ with the vector $\boldsymbol{\mathcal{O}}(\Theta)$ of the 2×2 hermitian Hamiltonians $H = \frac{1}{2}(\mathcal{O}_0 \mathbf{1} + \boldsymbol{\mathcal{O}}(\Theta) \cdot \boldsymbol{\sigma})$ and it is shown that the Hamiltonian H can be parameterized by two independent parameters $\mathcal{O}_0$ and Θ.

Keywords Quantum brachistochrone · Lower bound on time evolution · Hermitian quantum mechanics · Nonhermitian quantum mechanics

1 Introduction

Quantum system is governed by a Hamiltonian H and quantum states (we are considering only pure states here) of the system, belonging to a Hilbert space. The Hamiltonian acts on the states of this Hilbert space. In Schrödinger picture, the state evolves and it evolves in such a way that the norm of the state remains fixed, i.e., the evolution is unitary. Unitary evolution is known to be dictated by the unitary operator $\mathcal{U} = \exp(-itH)$ [1–3]. Here t is the evolution time of the system from an initial state to a final state of the system. In reality, in some situations, this evolution time has prime importance to think about. For example, in quantum computation it is desirable to minimize time of evolution of the orthogonal states of q-bits and it essentially depends on the transformation speed. The least time to transform

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one state to other orthogonal state is known to be $\Delta T = \pi/2E$ [4, 5], where E is the energy of the quantum system.

Research work in minimal time evolution has got lots of interest in recent years [4, 6–10], due to its applicability in quantum computation [11–16]. Quantum computation in least possible time is always desirable and can be achieved by using time optimal evolution of the quantum states. Although one can minimize time by designing the gates in a specific way, time optimal evolution is being thought of as an alternative.

The evolution of a system between two given states in minimum time is known to be quantum brachistochrone problem [7], a concept which has come from the brachistochrone problem in classical mechanics [17]. In hermitian quantum mechanics the minimal time to evolve a state to other state has a lower bound for a suitably chosen Hamiltonian provided the difference between two eigenstates E_1, E_2 $(E_2 > E_1)$ of the Hamiltonian is kept fixed, i.e., $E_2 - E_1 = 2\Delta E = constant$.

The search for further minimization of minimal time evolution naturally throws research work to the non-hermitian [18–34] quantum mechanical domain. The breakthrough is the recent remarkable work by Bender et al. They showed that with the same energy constraint $2\Delta E = constant$, the evolution time ΔT of a spin up state to the spin down state under a particularly chosen non-hermitian $\mathcal{PT}$-symmetric Hamiltonian $H_{\mathcal{PT}}$, can be made arbitrarily small. The reason behind this peculiar behavior can be understood from the fact that the states which are orthogonal in hermitian quantum mechanics under ordinary inner product, are non-orthogonal in non-hermitian quantum mechanics under $\mathcal{CPT}$ inner product. In fact using this $\mathcal{CPT}$ inner product, the distance between the two states can be made zero in non-hermitian quantum mechanics, although it has a finite distance in hermitian quantum mechanics. It can be noted that the shortest evolution time ΔT, the distance between two states ΔS and the energy difference $2\Delta E$ between the eigenvalues of the Hamiltonian, which can be considered as the speed of evolution of the system, can be related by $\Delta S = 2\Delta E \times \Delta T$ [35]. Because of the linear relation between the distance and evolution time, when the speed is fixed, one can achieve the evolution in arbitrarily short time if the distance can be made arbitrarily small.

This arbitrary short evolution time seems to be the result of introduction of the non-hermiticity and $\mathcal{PT}$-symmetry of the system. But if one sacrifices $\mathcal{PT}$-symmetry, is it still possible to get faster evolution than hermitian quantum theory? Although without $\mathcal{PT}$-symmetry, the non-hermitian system generates complex energy eigenvalues, it is however still possible to get arbitrarily short evolution [8], while keeping same the energy constraint, $2\Delta E = constant$.

Motivation for our present work comes from the previous works [7, 36, 37] on quantum brachistochrone problem in both hermitian and non-hermitian quantum mechanics. In this article we ask the most natural question regarding the optimal time evolution of quantum states. The question is; what is the minimal time ΔT to transform one state to another and then transform back to the same state? Answer to this question is so far not know for both hermitian and non-hermitian quantum mechanics. In this article we discuss this issue for both the cases, the hermitian and the non-hermitian quantum mechanics. We show that the minimal time required to transform one state to other and then transform back to the same initial state is the same, both for hermitian and non-hermitian quantum mechanics.

This article is organized as follows: In Sect. 2, we calculate minimal time to transform one state to its orthogonal state and then transform back to the same state in hermitian quantum mechanics. As a specific case, we consider the simple example of two state system (it can be a q-bit system or spin half system or other two state system), which has been

discussed in many places [5, 37–39]. In Sect. 3, we repeat the same calculation in non-hermitian $\mathcal{PT}$-symmetric quantum mechanical domain. We found that in both cases the minimal time ΔT of evolution is same. We conclude in Sect. 4.

2 Hermitian Quantum Mechanics (HQM)

In hermitian quantum mechanics, the generator of time evolution of a quantum system, which is the Hamiltonian itself, has to be hermitian $H = H^\dagger$ in order to make the evolution operator $\mathcal{U} = \exp(-itH)$ unitary with respect to the ordinary inner product in Hilbert space. We consider a two state system (it may be a q-bit, or a spin half system or other two state system), which can be described by a 2×2 hermitian matrix Hamiltonian. The Hilbert space of a two state system is the so called Bloch sphere [38]. Two dimensional projective Hilbert space, which is the set of one dimensional subspaces of complex two dimensional Hilbert space is isomorphic to the boundary of Bloch sphere S^2, which is the space of pure states of a two state system. Each pure state is identified with a point on this sphere S^2. Orthogonal states are identified with the antipodal points on S^2. So the general evolution of a state to another state (not necessarily orthogonal state) is identified as a curve on S^2. The geodesic, which is the shortest possible paths on S^2 between two given points, is given by the arch length (which should be part of the great circle through that two points). A generic pure quantum state vector can be written in terms of the density matrix $\rho = \frac{1}{2}(\mathbf{1} + \mathbf{P} \cdot \boldsymbol{\sigma})$ [39], where $\mathbf{P} \equiv (\mathbf{P_1}, \mathbf{P_2}, \mathbf{P_3})$ are real parameters, $\boldsymbol{\sigma} \equiv (\sigma_1, \sigma_2, \sigma_3)$ are Pauli matrices and $\mathbf{1}$ is the identity matrix. $\mathbf{P}$ is called the polarization vector with norm $P = \sqrt{\mathbf{P}^2} = 1$. General properties of these density matrix is that its trace is unity $\mathrm{Tr}(\rho) = 1$ and it is hermitian $\rho = \rho^\dagger$, which is evident from the expression of density matrix representation ρ.

Consider two orthogonal states (i.e., antipodal points on the sphere) designated by the two density matrix ρ (alternatively quantum state vector $|\Psi_I\rangle$) and $\tilde{\rho}$ (alternatively quantum state vector $|\Psi_F\rangle$) respectively. Our objective is to discuss the transformation of the state ρ to $\tilde{\rho}$ and then back to ρ in minimal time. Here, the geodesics are the great circles starting from the polar point ρ and passing through antipodal point $\tilde{\rho}$. It can be noted that the set of geodesics between two antipodal points ρ and $\tilde{\rho}$ can be parameterized by a parameter $\Theta \in \mathbb{R} \pmod{2\pi}$ of the unitary group $U(1) = \exp(i\Theta)$. It is thus expected that corresponding to each geodesic, characterized by a specific value of the parameter Θ, there exists a hermitian Hamiltonian $H(\Theta)$, which will transform the state ρ to $\tilde{\rho}$ and back to ρ. The total distance from the point designated by ρ to $\tilde{\rho}$ and then coming back to again ρ along the geodesic is the circumference of the great circle through that two point, which is $\Delta S_{\mathrm{HQM}} = 2\pi$. The maximum speed of transformation along the geodesic is $\Delta V_{\mathrm{HQM}} = E_2 - E_1 = 2\Delta E$. We now find the total minimal time ΔT_{HQM} of transformation for $\rho \to \tilde{\rho} \to \rho$ to be

$$\Delta T_{\mathrm{HQM}} = \Delta S_{\mathrm{HQM}} / \Delta V_{\mathrm{HQM}} = \pi / \Delta E. \tag{1}$$

We now need to find out the hermitian Hamiltonians $H_{\mathrm{HQM}}(\Theta)$, which correspond to the minimal time evolution (1) along the great circles. A general hermitian 2×2 matrix Hamiltonian on S^2 can be written in terms of four independent parameters $\mathcal{O}_0$ and $\boldsymbol{\mathcal{O}} \equiv (\mathcal{O}_1, \mathcal{O}_2, \mathcal{O}_3)$ as $H = \frac{1}{2}(\mathcal{O}_0 \mathbf{1} + \boldsymbol{\mathcal{O}} \cdot \boldsymbol{\sigma})$ [39], where $\mathbf{1}$ and $\boldsymbol{\sigma}$ has been defined already. In our case all four parameters will not be independent, because we need to impose some constraints on the Hamiltonian in order to achieve the desired Hamiltonian which will transform a given state to its orthogonal state and then back to the original state in minimal time. The general transformation of a state $\rho = \frac{1}{2}(\mathbf{1} + \mathbf{P} \cdot \boldsymbol{\sigma})$, can be associated with the

rate of change of the polarization vector $\mathbf{P}$ as $\frac{d\mathbf{P}}{dt} = \boldsymbol{\mathcal{O}} \times \mathbf{P}$ [39] (here $\times$ is the vector cross product). Obviously the rate of change will be maximum when the vector $\boldsymbol{\mathcal{O}}$ is perpendicular to the polarization vector $\mathbf{P}$. So, we identify two constraints for our Hamiltonian: (1) $2\Delta E = E_2 - E_1 = \mathcal{O}$, where $\mathcal{O}$ is the norm of the vector $\boldsymbol{\mathcal{O}}$. (2) $\boldsymbol{\mathcal{O}}$ is perpendicular to the plane of polarization vector through which it evolves. We also identify two arbitrariness involved in the Hamiltonian: (1) The sum of the two eigenvalues E_1 and E_2, i.e., $E_1 + E_2 = \mathcal{O}_0$ (this is arbitrary) and (2) The direction of the vector $\boldsymbol{\mathcal{O}}$, which is confined in a plane. So, now $\boldsymbol{\mathcal{O}}$ can be parameterized with a single parameter, which we identify with our previously defined parameter Θ. Then, we can immediately write down the Hamiltonian $H_{\mathrm{HQM}}(\mathcal{O}_0, \Theta)$, which will transform the state ρ to its orthogonal state in minimal time as

$$H_{\mathrm{HQM}}(\mathcal{O}_0, \Theta) = \frac{1}{2}(\mathcal{O}_0 \mathbf{1} + \boldsymbol{\mathcal{O}}(\Theta) \cdot \boldsymbol{\sigma}). \tag{2}$$

This Hamiltonian has only two independent parameters $\mathcal{O}_0$ and Θ as it should be. Each value of Θ corresponds to a transformation along a particular geodesic and for a fixed geodesic, characterized by $\Theta = \Theta_0$ the transformation can be achieved by a 1-parameter $\mathcal{O}_0 \in \mathbb{R}$ family of Hamiltonians $H_{\mathrm{HQM}}(\mathcal{O}_0, \Theta = \Theta_0)$.

3 Non-hermitian Quantum Mechanics (NQM)

As mentioned in the introduction, the search for much faster time evolution than hermitian quantum theory compels one to think beyond hermitian quantum domain. The obvious choice is the non-hermitian $\mathcal{PT}$-symmetric quantum mechanics [40–44], because it is a consistent quantum theory and can be thought of as an alternative to the conventional quantum mechanics. Although one may worry about the unitarity of time evolution, but it can be shown that under the newly defined inner product, the non-hermitian $\mathcal{PT}$-symmetric quantum mechanics respects unitarity. So one can develop quantum brachistochrone problem in this alternative quantum mechanical setting. Here, our objective is to calculate the minimal time evolution ΔT_{NQM} from a state ρ (alternatively quantum state vector $|\Psi_I\rangle$) to its orthogonal state (according to hermitian quantum mechanics) $\tilde{\rho}$ (alternatively quantum state vector $|\Psi_F\rangle$) and then back to the same state ρ.

But before going into our objective, let us review why in non-hermitian $\mathcal{PT}$-symmetric theory, transformation between two orthonormal states (orthonormal according to hermitian theory) can be made in arbitrarily short time [37]. The clue is in the nontrivial $\mathcal{CPT}$ inner product, defined for the non-hermitian $\mathcal{PT}$-symmetric quantum mechanics in order to make the theory unitary. The crucial difference between the ordinary inner product, defined over Hilbert space in HQM and $\mathcal{CPT}$ inner product defined over Hilbert space in NQM is that the orthogonal states in HQM becomes non-orthogonal in NQM. This in effect changes the distance $\Delta S_{\mathrm{NQM}} = 2\cos^{-1}(|\langle\Psi_F|\Psi_I\rangle|)$ [37, 45] between the state $|\Psi_I\rangle$ and $|\Psi_F\rangle$. In hermitian quantum mechanics, this distance ΔS_{HQM} between two orthogonal states become $\Delta S_{\mathrm{HQM}} = \pi$, but in non-hermitian quantum mechanics, the distance between these same two states become $\Delta S_{\mathrm{NQM}} = \pi - 2|\alpha|$ [37], where α is a real parameter dependent on the Hamiltonian H_{NQM} of the corresponding non-hermitian quantum theory. One can choose a Hamiltonian H_{NQM} for NQN in such a way that $|\alpha| \to \pi/2$, thereby making the distance $\Delta S_{\mathrm{NQM}} \to 0$. This nontrivial property of the NQM has been capitalized in [37] to show that the minimal time to transform a spin up state to a spin down state is $\Delta T_{\mathrm{NQM}} = \Delta S_{\mathrm{NQM}}/\Delta V_{\mathrm{NQM}} = 0$, where ΔV_{NQM}, the speed of the transformation is kept

fixed throughout our analysis, both in hermitian and non-hermitian quantum mechanics i.e., $\Delta V_{\mathrm{NQM}} = \Delta V_{\mathrm{HQM}} = 2\Delta E$.

Now we return to our objective, which is to calculate the total minimal time to transform a state back to its initial state through the orthonormal (orthonormal in HQM) state. In order to calculate that, we need to calculate the distance from a state to its orthonormal state (orthonormal in HQM) and then back to the initial state. This total distance is $\Delta S_{\mathrm{NQM}} = 2\pi$, which is independent of the parameter α. This shows that in both cases, hermitian and non-hermitian, the total distance is same. In fact this total distance is the circumference of the unit radius circle on S^2. We now easily calculate the minimal total time of transformation to be

$$\Delta T_{\mathrm{NQM}} = \Delta S_{\mathrm{NQM}}/\Delta V_{\mathrm{NQM}} = \pi/\Delta E. \tag{3}$$

It is evident from (1) and (3) that in both hermitian and non-hermitian quantum mechanics, the total minimal time to transform a state to its orthonormal state and then back to the initial state, is same. Here we need to clarify our result with respect to the result of reference [37]. It can be easily understood that our result is in agreement with [37]. Divide the total distance ΔS_{NQM} for the transformation $\rho \to \tilde{\rho} \to \rho$ into two parts. One $\Delta S_{\mathrm{NQM1}} = \pi - 2|\alpha|$ for the transformation $\rho \to \tilde{\rho}$ and other $\Delta S_{\mathrm{NQM2}} = \pi + 2|\alpha|$ for the transformation $\tilde{\rho} \to \rho$. If we make $\Delta S_{\mathrm{NQM1}} = 0$ by tuning $|\alpha| \to \pi/2$, then the other distance becomes $\Delta S_{\mathrm{NQM2}} = 2\pi$. The total distance however remains fixed i.e., $\Delta S_{\mathrm{NQM}} = \Delta S_{\mathrm{NQM1}} + \Delta S_{\mathrm{NQM2}} = 2\pi$ and total minimal time thus has a lower bound $\pi/\Delta E$.

4 Conclusion and Discussion

We have shown that to make a round trip along the great circle over S^2, the Hamiltonian needs a minimal time which has a lower bound $\Delta T = \pi/\Delta E$ in quantum mechanics. Non-hermitian quantum mechanics can not reduce it to arbitrarily small value. We have also shown that the total minimal time is same in both quantum mechanics. Our result is shown to be in agreement with the recent work [37] on quantum brachistochrone problem. We have also shown that in hermitian quantum mechanics the condition for minimal time evolution can be considered as a constraint coming from the orthogonality of the polarization vector $\mathbf{P}$ of the evolving quantum state $\rho = \frac{1}{2}(\mathbf{1} + \mathbf{P}\cdot\boldsymbol{\sigma})$ with the vector $\boldsymbol{\mathcal{O}}(\Theta)$ of the 2×2 hermitian Hamiltonians $H = \frac{1}{2}(\mathcal{O}_0\mathbf{1} + \boldsymbol{\mathcal{O}}(\Theta)\cdot\boldsymbol{\sigma})$. The Hamiltonian H can be parameterized by two independent parameters $\mathcal{O}_0$ and Θ. $\mathcal{O}_0$ has been identified as the sum of the eigenvalues of the Hamiltonian and Θ is responsible for evolution along different great circles through the two given antipodal points. After we submitted our work to arXiv others also reported in this subject [46–51].

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